Numerical Solution of Stefan-Like Problems, Theory and Algorithm

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Contents

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One-Phase Stefan Problem

The domain $\Omega \subset R^2$, $\Omega \equiv \{(x,t) |\, 0 < x < s(t), 0 < t < \infty\}$ and $\Gamma \equiv \{(s(t),t) |\, 0 \le t < \infty\}$. Find $\{u(x,t), s(t)\}$, where $u \in C^{(2,1)}$ and $s \in C^{(1)}$, such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for all} \quad (x,t) \in \Omega, \tag{1.4}$$

$$s(0) = 0, \quad u(0,t) = -1, \quad u(s(t),t) = 0, \quad t > 0, \tag{1.5}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=s(t)^-} = L\frac{\mathrm{d}s(t)}{\mathrm{d}t}, \quad t > 0, \tag{1.6}$$

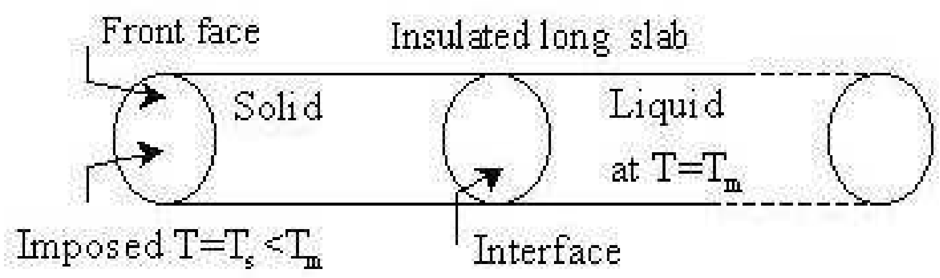

Figure 2. 2.1. Physical realization of the one-phase Stefan problem.

Numerical Solution of MBPs

- **A Mathematical Problem.** Find the triple $\{u_1(x,t), u_2(x,t), s(t)\}$, for which

$$L_i u_i \equiv \Big(\frac{\partial}{\partial x}\big(p_i(x,t)\frac{\partial}{\partial x}\big) + a_i(x,t)\frac{\partial}{\partial x} - b_i(x,t) - d_i(x,t)\frac{\partial}{\partial t}\Big)u_i = f_i(x,t),$$

$$(x,t) \in \Omega_i, \quad t > 0, \quad i = 1, 2, \tag{2.1}$$

where Ω_i is the subset of the rectangle $(l_1, l_2) \times (0, K)$ such that

$$(x,t) \in \Omega_1 \Leftrightarrow t \in (0,K) \wedge x \in (l_1, s(t)) \equiv Q_1(s(t)),$$

and

$$(x,t) \in \Omega_2 \Leftrightarrow t \in (0,K) \wedge x \in (s(t), l_2) \equiv Q_2(s(t)),$$

$s(t)$ is the moving boundary, and $K < \infty$ is some arbitrary but fixed upper limit.

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Numerical Solution of MBPs

Fixed boundary and initial conditions are,

$$\alpha_i(t)u_i(l_i,t) + (-1)^i\beta_i(t)p_i(l_i,t)\frac{\partial u_i}{\partial x}(l_i,t) = \gamma_i(t), \quad t>0, i=1,2 \tag{2.2}$$

$$s(0) = s^0, \quad l_1 \le s^0 \le l_2, \tag{2.3}$$

$$u_1(x,0) = u_1^0(x), \quad l_1 \le x \le s^0, \quad u_2(x,0) = u_2^0(x), \quad s^0 \le x \le l_2. \tag{2.4}$$

where, α_i, β_i, γ_i, u_i^0, $i=1,2$, are given functions. We suppose that $\alpha_i(t) \ge 0$, $\beta_i(t) \ge 0$, $\alpha_i(t) + \beta_i(t) \ne 0$, $i=1,2$, $t>0$.
Conditions at the moving interface $x = s(t)$ will be written as

$$H\Big(u_1(s,t), u_2(s,t), p_1(s,t)\frac{\partial u_1}{\partial x}(s,t), p_2(s,t)\frac{\partial u_2}{\partial x}(s,t), s(t), \frac{ds(t)}{dt}, t\Big) = 0, \quad t>0, \tag{2.5}$$

where $H = (H_1, H_2, H_3)$ is a given function with values in R^3.

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Numerical Solution of MBPs

We first apply the method of time discretization called also Rothe's method. Using implicit Euler discretization in time, the resulting approximate free boundary (interface) problem at the time level $t = t^n$ may be written as

$$[p_i^n(x) u_i^{n\prime}]' + a_i^n(x) u_i^{n\prime} - b_i^n(x) u_i^n - d_i^n(x) \frac{u_i^n - \hat{u}_i^{n-1}(x)}{\Delta t} = f_i^n(x),$$

$$x \in Q_i(s^n), \quad i = 1, 2, \tag{2.9}$$

$$\alpha_i^n u_i^n(l_i) + (-1)^i \beta_i^n p_i^n(l_i) u_i^{n\prime}(l_i) = \gamma_i^n, \quad i = 1, 2, \tag{2.10}$$

$$H^n\left(u_1^n(s^n), u_2^n(s^n), p_1^n(s^n) u_1^{n\prime}, p_2^n(s^n) u_2^{n\prime}, s^n, \frac{s^n - s^{n-1}}{\Delta t}, t^n\right) = 0, \tag{2.11}$$

Our task now is to find the triples $\{u_1(x), u_2(x), s\}$ satisfying (2.9)–(2.11) at successive times t^n, $n = 1, \ldots, N$.

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Numerical Solution of MBPs

To solve the unknown interface problem at one time level we apply method of transfer of conditions by J. Taufer. For $\alpha_i > 0$ from the **Theorem 2.1**(discussed in the paper) we solve

$$y_i^{n\,\prime} = \Big[\frac{d_i^n(x)}{\Delta t} + b_i^n(x)\Big] y_i^{n\,2} + \frac{a_i^n(x)}{p_i^n(x)} y_i - \frac{1}{p_i^n(x)} \quad a.e. \quad \text{on} \quad [l_1, l_2], \tag{2.14}$$

$$y_i(l_i) = (-1)^i \frac{\beta_i^n}{\alpha_i^n}, \quad i = 1, 2,$$

$$z_i^{n\,\prime} = \Big[\frac{d_i^n(x)}{\Delta t} + b_i^n(x)\Big] y_i^n(x) z_i^n - \Big[\frac{\hat{u}_i^{n-1}(x)}{\Delta t} - f_i^n(x)\Big] y_i^n(x) \quad a.e. \quad \text{on} \quad [l_1, l_2], \tag{2.15}$$

$$z_i^n(l_i) = \frac{\gamma_i}{\alpha_i}, \quad i = 1, 2.$$

The functions y_i, z_i possess the property that any absolutely continuous function u_i that satisfies ODE a.e. and for which fixed boundary condition holds satisfies also the transferred condition

Numerical Solution of MBPs

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$$u_i^n(x) = z_i^n(x) - y_i^n(x) p_i^n(x) u_i^{n\prime}(x) \quad \forall x \in [l_1, l_2], \quad i = 1, 2. \tag{2.16}$$

If $y_i^n(x) \neq 0$ we may express $p_i^n(x) u_i^{n\prime}$ from (2.16) and substitute into the unknown interface condition:

$$H^n\left[u_1^n(s^n), u_2^n(s^n), \frac{z_1^n(s^n) - u_1^n(s^n)}{y_1^n(s^n)}, \frac{z_2^n(s^n) - u_2^n(s^n)}{y_2^n(s^n)}, s^n, \frac{s^n - s^{n-1}}{\Delta t}, t^n\right] = 0. \tag{2.23}$$

The transferred condition (2.16) is sometimes called *Riccati transformation* since (2.14) is the *Riccati equation*.

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Properties of Algorithm

We learn the properties of the algorithm and work on the following two problems:

- The feasibility of algorithm.
- The existence of the interface s^n.

An ordinary differential equation of the form

$$y'(x) = A(x)y^2(x) + B(x)y(x) + C(x) \tag{3.1}$$

is known as a *Riccati equation or a generalized Riccati equation*
Let y and ω be such that $y(x) = \frac{\omega(x)}{e^{-\int_0^x B(\xi)d\xi}} = \frac{\omega(x)}{E(x)}, \quad x \in [0,1]$ then the **Lemma 3.1** (in the paper) enables us to study the following equation instead of (3.1)

$$\omega' = P(x)\omega^2 + Q(x), \quad \forall x \in (0,1), \quad \omega(0) = y_0. \tag{3.4}$$

In addition, $\operatorname{sgn} P(x)$=$\operatorname{sgn} A(x)$, $\operatorname{sgn} Q(x)$=$\operatorname{sgn} C(x)$, $\operatorname{sgn} \omega(x)$= $\operatorname{sgn} y(x)$.

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Properties of Algorithm

Theorem 3.3. Let $P, Q \in C([0,1])$ and $P(x) \geq 0, Q(x) \leq 0 \; \forall x \in [0,1]$. Let also $\omega_0 \leq 0$ and $\omega_0 + Q(0) < 0$. Then there exists a unique continuous function $\omega : [0,1] \to R$ which satisfies the equation (3.4) on $[0,1]$ with the initial condition $\omega(0) = \omega_0$ and furthermore $\omega(x) < 0 \; \forall x \in (0,1]$.

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- **Development on One-Phase Stefan Problem**

$$Lu \equiv u_{xx} + a(x,t)u_x - b(x,t)u - d(x,t)u_t = f(x,t), \quad (x,t) \in \Omega_0, \tag{3.18}$$

where $\Omega_0 = \{(x,t) : 0 < x < s(t), t > 0\}$. The conditions at the moving interface are

$$u(s(t),t) = 0, \quad t > 0, \tag{3.20a}$$

$$\frac{ds}{dt} + k(s(t),t)u_x(s(t),t) = \eta(s(t),t), \quad t > 0. \tag{3.20b}$$

Properties of Algorithm

Assumption 3.1

1. All the functions in the equation (3.18) and in the condition (3.20b) are continuous and bounded on $[0, \infty) \times [0, K]$, the function α is continuous on $[0, K]$ and the function u_0 is continuous on $[0, s^0]$.
2. We suppose that $\alpha(t) > 0$, $t \in [0, K]$; $k(x, t) \geq 0$, $\eta(x, t) \geq 0$, $0 \leq \underline{a} \leq a(x, t) \leq \overline{a}$, $0 \leq \underline{b} \leq b(x, t) \leq \overline{b}$, $0 \leq \underline{d} \leq d(x, t) \leq \overline{d}$, $f(x, t) \leq 0$, $(x, t) \in [0, \infty) \times [0, K]$; $u_0(x) \geq 0$, $x \in [0, s(0)]$, and $\alpha(0) = u_0(0)$.
3. There exist constants $M \geq 0$ and $\beta \geq 0$ such that
$$u_0(x) \leq M \frac{s^0 - x}{s^0}, \quad f(x, t) \geq \underline{b} M e^{\beta t} \frac{x - s^0}{s^0}, \quad x \in [0, s^0], \ t \in [0, K].$$
4. $f(x, t) = 0$, $x \geq s^0$, $t \in [0, K]$.
5. Functions $\eta(x, t)$, $k(x, t)$ are continuously differentiable on $[0, \infty) \times [0, K]$.

Existence of Solution

Using the time discretization method with the time step Δt, $n = 1, 2, ..., N$, the approximate problem for (3.18)–(3.20) is

$$L^n u^n \equiv u^{n\prime\prime} + a^n(x)u^{n\prime} - [b^n(x)u^n + \frac{d^n(x)}{\Delta t}]u^n =$$

$$= f^n(x) - \frac{d^n(x)}{\Delta t}\hat{u}^{n-1}(x), \quad 0 < x < s^n, \tag{3.21}$$

$$u^n(0) = \alpha^n, \tag{3.22}$$

$$u^n(s^n) = 0, \quad s^n - s^{n-1} + \Delta t k^n(s^n)u^{n\prime}(s^n) = \Delta t \eta^n(s^n). \tag{3.23}$$

Theorem 3.5. Let u^{n-1}, s^{n-1} be given and $u^{n-1}(x) \geq 0$, $x \in [0, s^{n-1}]$. If the problem (3.21)–(3.23) has a solution $\{u^n, s^n\}$, then $u^n(x) \geq 0$ for $x \in [0, s^n]$ and $s^n \geq s^{n-1}$.

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Existence of Solution

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Theorem 3.6. The free boundary s^n of (3.21)–(3.23) at $t = t^n$, if it exists, is a root of the equation $\varphi^n(x) = 0$. Conversely, any zero of φ^n in $(0, \infty)$ represents an admissible free boundary s^n of the problem (3.21)–(3.23).

$$\varphi^n(x) = x - s^{n-1} - \Delta t \left[\eta^n(x) - k^n(x) \frac{z^n(x)}{y^(x)} \right]. \tag{3.28}$$

Theorem 3.7. Given a couple $\{u^{n-1}, s^{n-1}\}$ such that $s^{n-1} \geq s^0$, $u^{n-1}(x) \geq 0$ for $x \in [0, s^{n-1}]$, there exists a solution $\{u^n, s^n\}$ of the free boundary problem (3.21)–(3.23) and any such solution satisfies $u^n(x) \geq 0$, $x \in [0, s^n]$, and $s^n \geq s^{n-1}$.

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Uniqueness of Solution

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Lemma 3.2. Let M and β be the constants from Assumption 3.1-3. Put

$$\|\alpha\| = \max_{[0,K]} \mid \alpha(t) \mid, \quad M_1 = \max(M, \|\alpha\|).$$

Suppose that

$$0 \leq u^{n-1}(x) \leq M_1 \frac{s^{n-1} - x}{s^{n-1}} e^{\beta(n-1)\Delta t}, \quad x \in [0, s^{n-1}].$$

Then we have

$$0 \leq u^n(x) \leq M_1 \frac{s^n - x}{s^n} e^{\beta n \Delta t}, \quad x \in [0, s^n]$$

for every solution u^n, s^n.

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Uniqueness of Solution

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Theorem 3.8. Let $\|\eta\|, \|k\|$ be constants such that $|\eta(x,t)| \leq \|\eta\|$, $|k(x,t)| \leq \|k\|$, $x \in [0,\infty)$, $t \in [0,K]$. Let β be the constant from Assumption 3.1-3 and let M_1 be the constant from Lemma 3.2. Set

$$\hat{M}_1 = M_1 e^{\beta K}, \quad C = \|\eta\| + \|k\| \frac{\hat{M}_1}{s^0}.$$

Then any solution $\{u^n, s^n\}$ of the problem (3.21)–(3.23) satisfies

$$0 \leq u^n(x) \leq \hat{M}_1, \quad x \in [0, s^n], n = 0, 1, ..., N, \tag{3.31}$$

$$0 \leq s^n - s^{n-1} \leq C\Delta t, n = 1, 2, ..., N, \tag{3.32}$$

and thus

$$0 \leq s^n \leq s^0 + CK, \quad n = 1, 2, ..., N. \tag{3.33}$$

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Uniqueness of Solution

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Theorem 3.9. For a sufficiently small Δt, the problem (3.21)–(3.23) has a unique solution $\{u^n, s^n\}$.

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$$\varphi' \geq 1 - \Delta t \|\eta_x\| - \|k_x\| \sqrt{\Delta t} e^{\overline{a} s^0} \sqrt{U} \left[\tanh \left(\sqrt{\frac{U}{\Delta t}} s^0 \right) \right]^{-1} \cdot \overline{M}_1 \qquad (3.39)$$

for $x \in [s^{n-1}, s^{n-1} + C]$. Hence, for sufficiently small Δt we obtain the estimate

$$\varphi^{n\prime}(x) > 0, \quad x \in [s^{n-1}, s^{n-1} + C],$$

and thus the function $\varphi^n(x)$ is increasing on $[s^{n-1}, s^{n-1} + C]$ and has at most one zero.

The Numerical Experiments

Problem. A slab, $0 \leq x \leq 1$, initially solid at temperature $T_s = u_2(x,0) = -0.5°$C and $u_1(x,0) = 0°$C (just formally), $s(0) = 0$, is melted from the left by imposing a temperature $T_l = u_1(0,t) = 1°$C at the face $x = 0$ and at the back face $x = 1$ use the Neumann temperature itself. Find the triple $\{u_1(x,t), u_2(x,t), s(t)\}$, where u_1 and u_2 denote the temperature in the liquid and solid phases, respectively.

Location of the free boundary $s(t)$ computed by two different algorithms

t	*Alg1*	*RE1*	*Alg2*	*RE2*	True Solution
4.0309	0.0351	0.0140	0.0346	0.0281	0.0356
6.0460	0.0461	0.0573	0.0447	0.0252	0.0436
8.0470	0.0523	0.0398	0.0505	0.0040	0.0503

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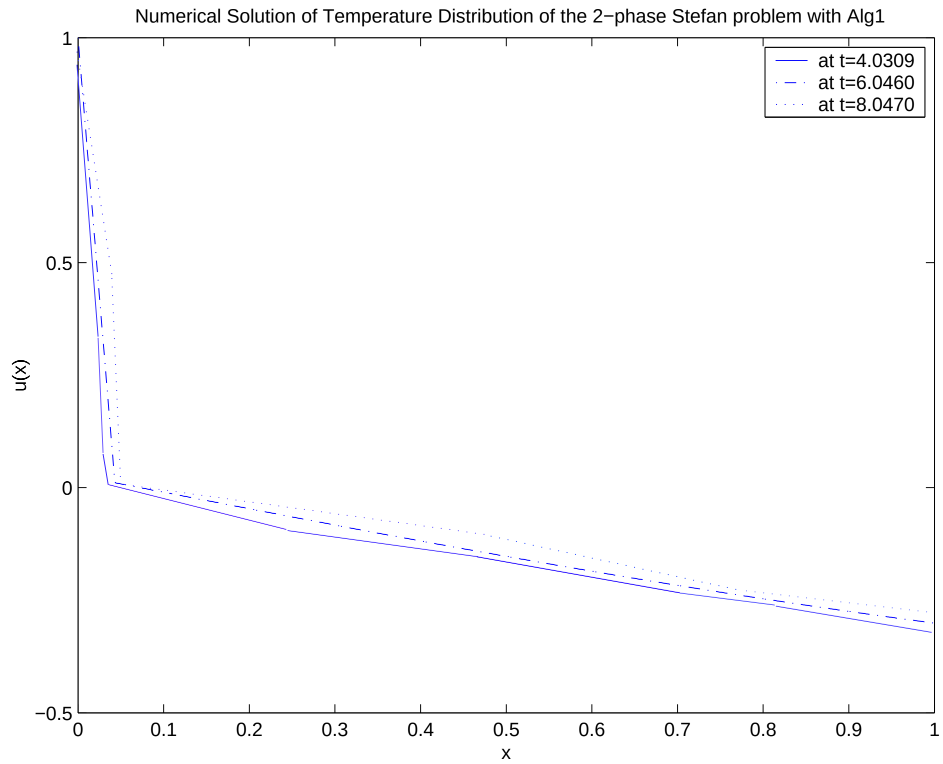
The Numerical Experiments



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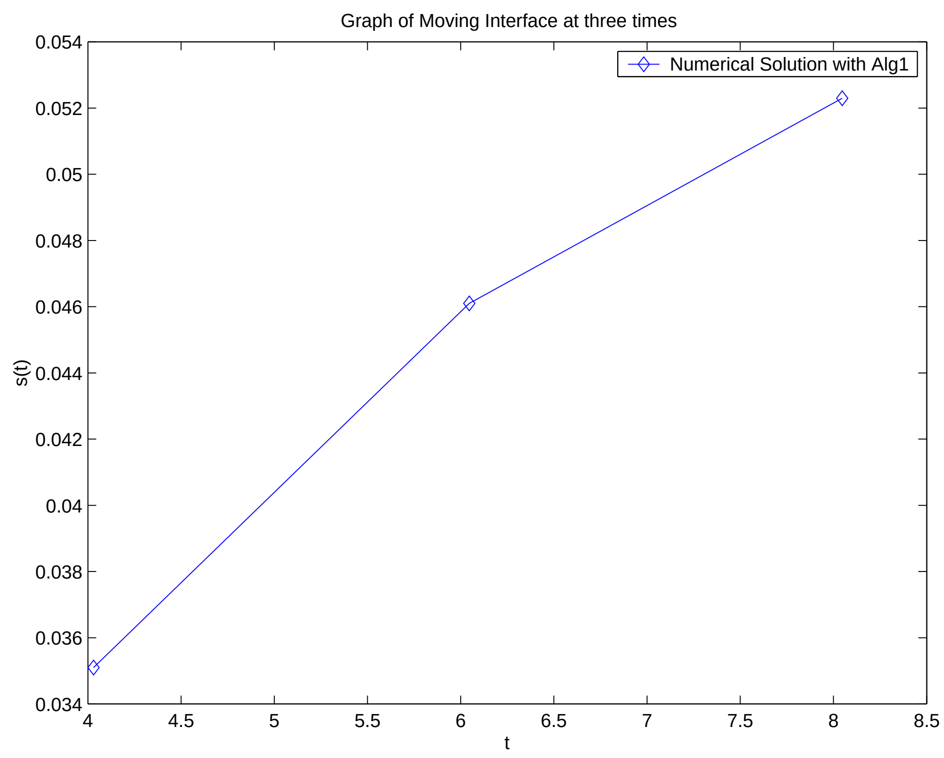

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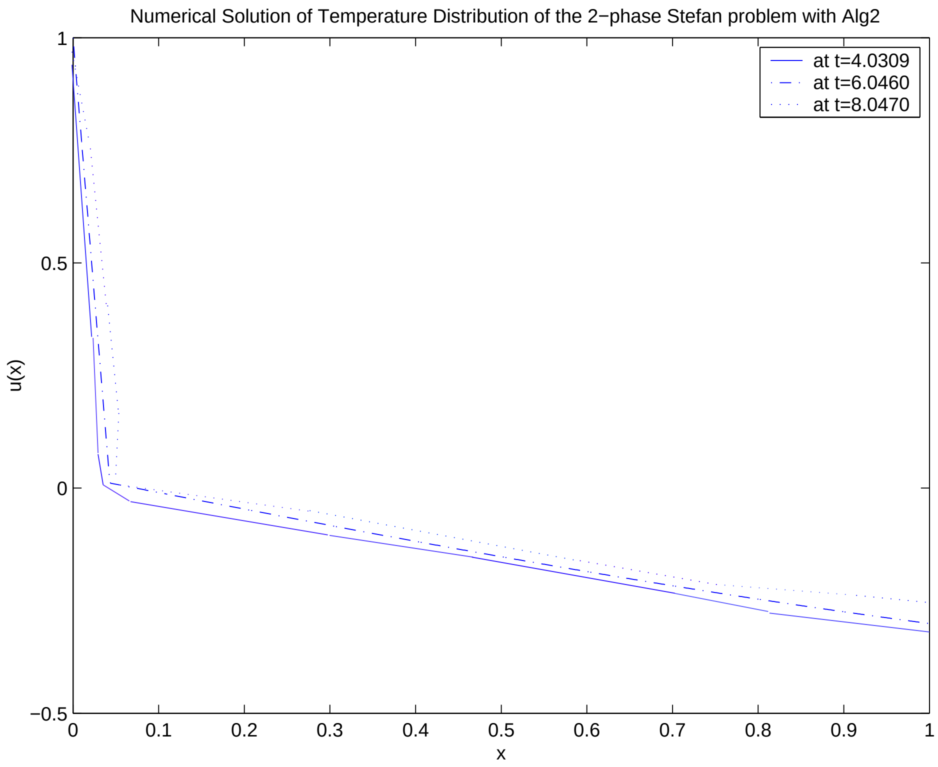

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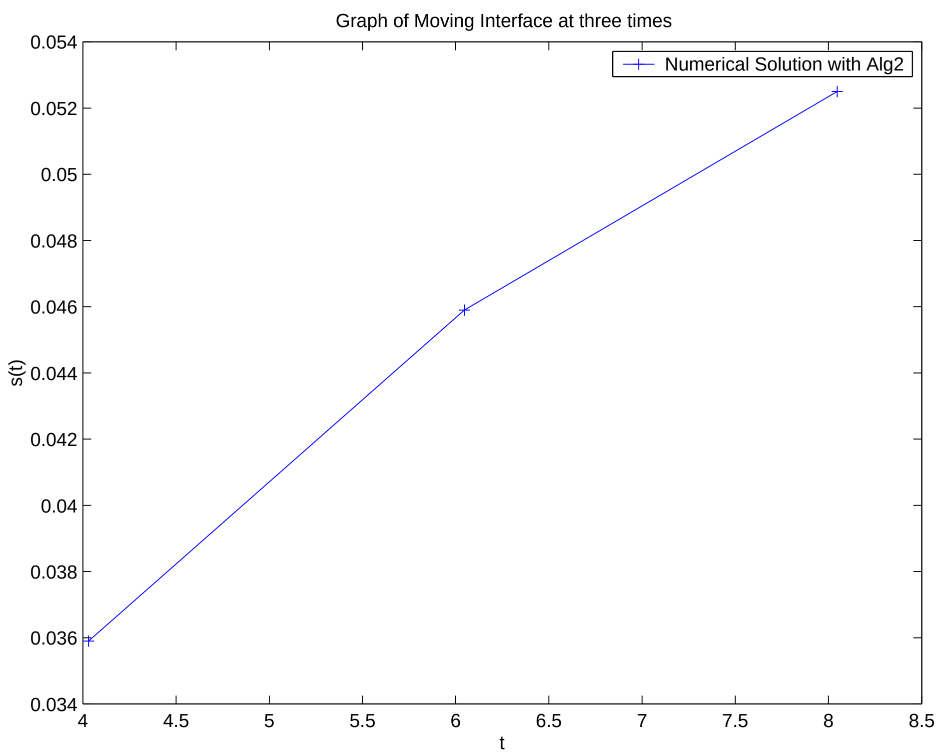

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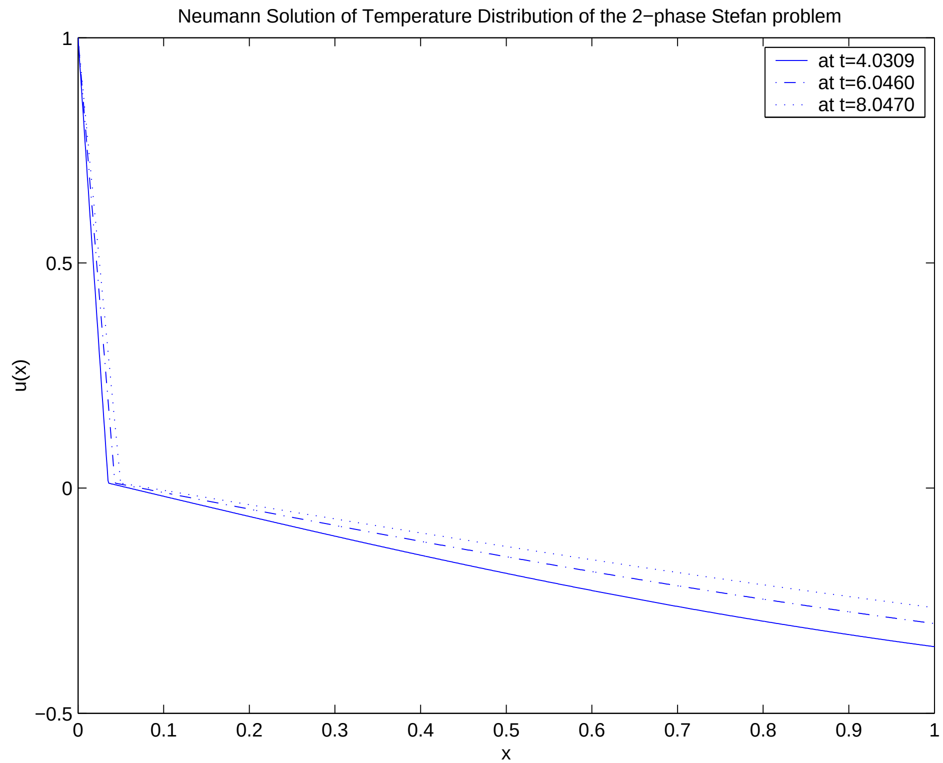

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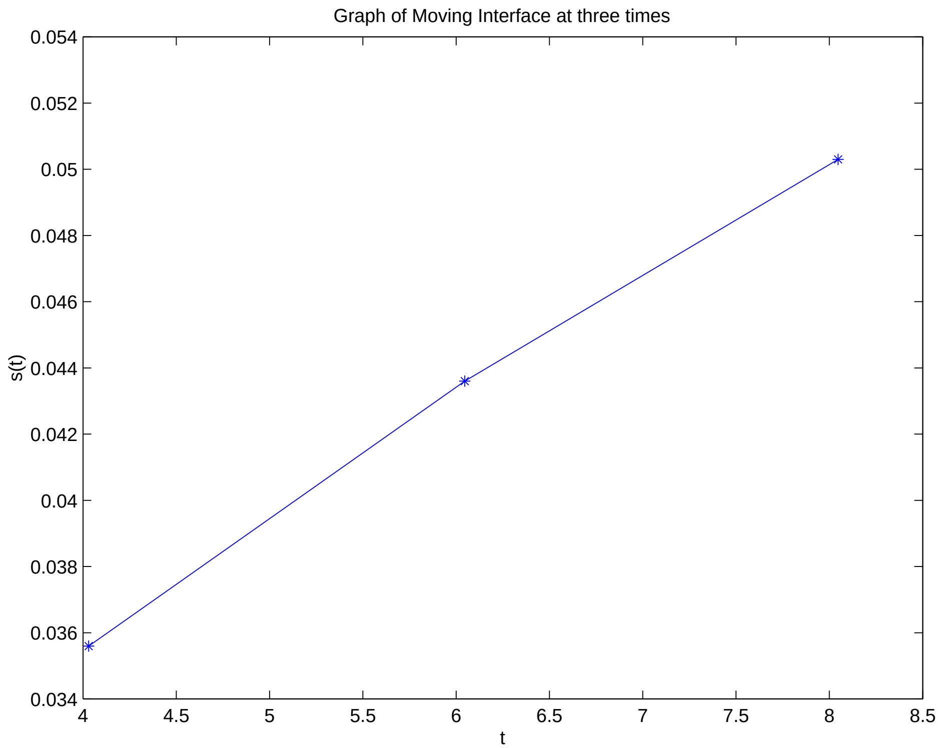

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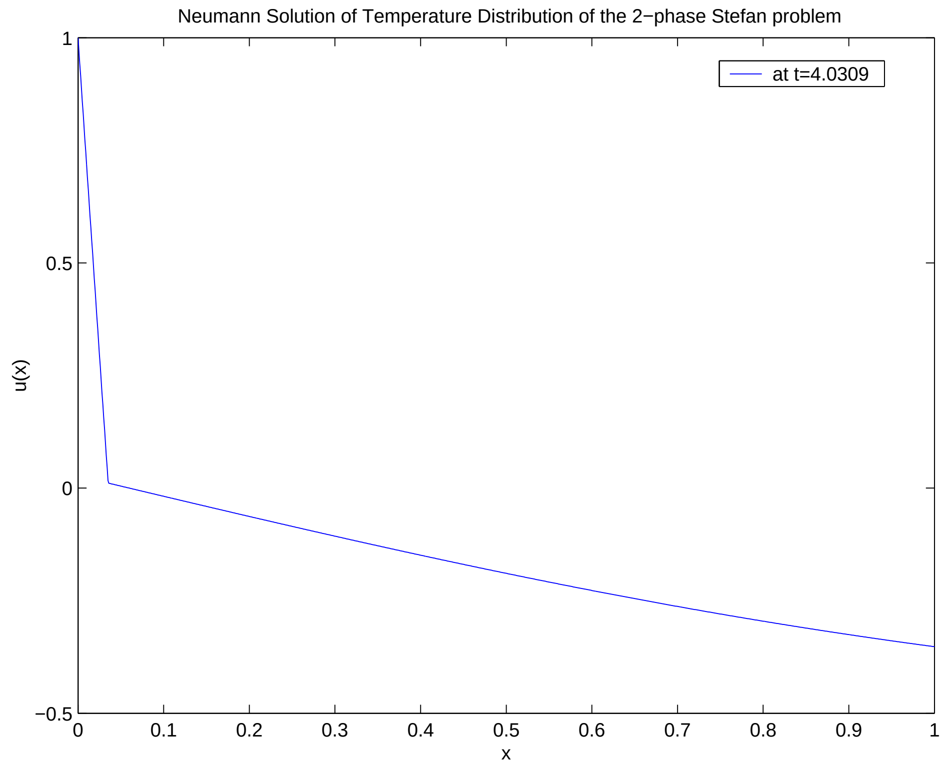

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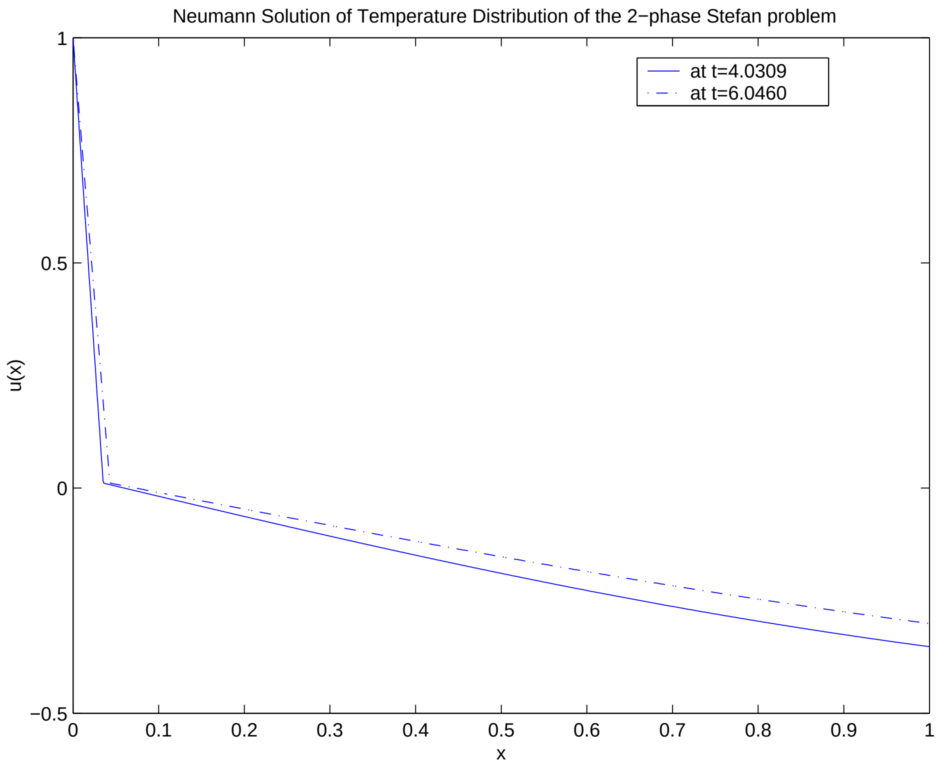

The Numerical Experiments